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Stability of an elastic cytoskeletal tensegrity model

K.A. Lazopoulos *

Mechanics Laboratory, Faculty of Applied Sciences (SEMFE), National Technical University of Athens, 5 Heroes of Polytechnion Avenue, Zografou campus, Athens, GR 157 73, Greece

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Abstract

An elastic cytoskeletal tensegrity structure composed by six inextensible elastic struts and 24 elastic cables is considered. The model is studied, adopting delay convention for stability. Critical conditions for simple and compound instabilities are defined. Post-critical behavior is also described. Equilibrium states with buckling of the struts are also considered. It is revealed that critical Euler buckling load of the struts is a necessary but not a sufficient condition for the existence of bifurcated equilibrium states, caused by buckling of the struts.

1. Introduction

Following the pattern of Snelson's sculptures in 1948, Buckminster Fuller (1961) adopted the concept of tensegrity as a new method of designing geodesic structures. Those structures are made up by struts, preserving the structural integrity by cables in tension. According to Connelly and Back (1998), the tension-integrity, or tensegrity, structures can be mathematically modeled as a configuration of points, or vertices, satisfying simple distance constraints. Ingber (1993, 1998), using tensegrity structures for modeling the cell deformability, proposed a microstructural approach to cytoskeletal mechanics based on tensegrity. In fact tensegrity structures are "strut-cable" structures with prescribed symmetries. There exists an extensive literature concerning the mechanics and also the advanced mathematics involved in these structures. A concise description of the topic, trying to put together the mathematics and the mechanics of tensegrity as well, has been given by Williams (2003) with up-dated references. Linear dynamic analysis results has been reported by Motro (1992). Nonlinear dynamics and control studies have been published by Sultan et al. (2001). In addition Coughlin and Stamenovic (1997) presented a study of a six strut tensegrity structure

* Tel.: +30 2294075816; fax: +30 2107721682.
E-mail address: kolazop@central.ntua.gr

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with buckling compression elements, applied to cell mechanics. Yet Stamenovic and Coughlin (1999) deal with the role of prestress and the architecture of the cytoskeleton presenting also some specific values of various elastic moduli. Since those structures are strong and light they have been quite important in the construction of huge structures, like domes, high structures such as antennas and deployable structures as well, used in space.

The simplest three dimensional tensegrity structure, classified as T-3 by Kenner (1976), has been studied by Oppenheim and Williams (1997, 2002, 2001). Following a bifurcation method, working with generalized coordinates, Lazopoulos (submitted for publication) studied the stability of the model. Further, Coughlin and Stamenovic (1997) studied the stability of the six-strut model, for microstructural cytoskeletal studies, Ingber (1993, 1998). Nevertheless, the last stability studies are restricted to the buckling of the struts.

In the present work, the stability of a six-strut tensegrity model is discussed with elastic inextensible struts. Euler-strut buckling is also considered. In fact the stress–strain laws of the cables are non-linear and non-unique equilibrium solutions show up. Critical states are defined and post-critical equilibrium paths are described adopting delay convention for stability, Gilmore (1981, p. 143) . The local (delay) convention for stability is invoked for buckling phenomena, whereas the global (Maxwell's) convention is mainly recalled for coexistence of phases phenomena such as elastoplastic ones, twinning of crystals, smart materials etc, Ericksen (1991), Pitteri and Zanzotto (2003). Simple and compound bifurcation of the equilibrium paths are revealed. Since the present bifurcation is a multivariable problem, standard methods require elimination of passive coordinates and normalization of the potential energy function (Thompson and Hunt, 1973; Troger and Steindl, 1991). Nevertheless, using a free coordinate bifurcation procedure (Lazopoulos, 1994), simple and compound bifurcation problems may be studied and the corresponding singularities may be classified. Some applications will be worked out just for implementation of the theory. Furthermore, critical states and post-critical equilibrium paths are considered caused by Euler's buckling of the struts. It is revealed that the critical Euler buckling load of the struts is a necessary condition for the critical states, however it does not always yield stable post-critical states.

2. Description of the six-strut tensegrity model

As it has been mentioned earlier, stability of the six-strut tensegrity model has been studied by Coughlin and Stamenovic (1997). Euler-strut buckling has already been discussed. The model shown in Fig. 1 consists of six inextensible struts and 24 cable segments. The cables and the struts are connected through joints. The origin O of the coordinate system $OXYZ$ is placed at the center of the model with the axes in the direction of the pairs of the struts. All the struts have the same initial length L_0. Nevertheless, the struts may buckle and their chords may change and become L_{I}, L_{II}, L_{III} correspondingly for the struts (AA or $A'A'$), (BB or $B'B'$), (CC or $C'C'$). Furthermore, the initial (without external loading) length of the cables is equal to $l_0 = \sqrt{3/8}L_0$. The value of the length l_0 is required for the equilibrium conditions of the structure without external loads. Forces of magnitude $T_x/2$ are applied at the end points of the struts AA and $A'A'$, see Fig. 1, while forces of magnitude $T_y/2$ are applied at the ends of the struts BB and $B'B'$. Yet forces of magnitude $T_z/2$ are applied at the ends of the struts CC and $C'C'$. This causes the change of the distances between the struts. In fact the distance between the AA and $A'A'$ struts changes from $s_0 = \frac{L_0}{2}$ to s_x. The same holds for the distance s_y between the BB and $B'B'$ struts and s_z between the CC and $C'C'$ struts. Furthermore, the cable segments change from l_0 to l_1 for the segments AB, $A'B$, AB', $A'B'$, and from l_0 to l_2 for the segments AC, $A'C$, AC', $A'C'$. Likewise the cable segments BC, $B'C$, BC', $B'C'$ change from l_0 to l_3. Hence the expressions for the current cable segments l_1, l_2, l_3 are given by, see Coughlin and Stamenovic (1997),

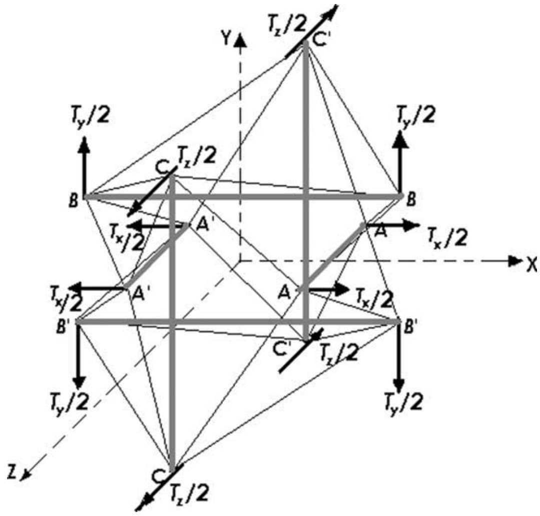

Fig. 1. The geometry of the six-strut tensegrity model.

$$
\begin{aligned}
l_1 &= \frac{1}{2}\sqrt{(L_{\mathrm{II}} - s_x)^2 + s_y^2 + L_{\mathrm{I}}^2},\\
l_2 &= \frac{1}{2}\sqrt{(L_{\mathrm{I}} - s_z)^2 + s_x^2 + L_{\mathrm{III}}^2},\\
l_3 &= \frac{1}{2}\sqrt{(L_{\mathrm{III}} - s_y)^2 + s_z^2 + L_{\mathrm{II}}^2},
\end{aligned} \tag{1}
$$

with the current chord lengths L_{I}, L_{II}, L_{III}. The analysis concerning buckling of the struts will be worked out in the section following the stability study of the tensegrity model with inextensible struts.

Let us consider a non-linear strain energy function for the cables,

$$
W_i = \frac{\alpha}{2}\left(\frac{l_i}{l_{\mathrm{n}}} - 1\right)^2 + \frac{\beta}{6}\left(\frac{l_i}{l_{\mathrm{n}}} - 1\right)^3 + \frac{c}{24}\left(\frac{l_i}{l_{\mathrm{n}}} - 1\right)^4, \tag{2}
$$

with $i = 1, 2, 3$ and l_{n} the initial natural lengths of the cable. Then the total potential energy function with rigid struts is given by

$$
V = 8(W_1 + W_2 + W_3) - T_x s_x - T_y s_y - T_z s_z. \tag{3}
$$

Hence, the total potential energy function depends on the variables s_x, s_y, s_z and the loading parameters T_x, T_y, T_z. Thus the equilibrium equations are given by

$$
\mathbf{V}_1 = \nabla V = \left(\frac{\partial V}{\partial s_x}, \frac{\partial V}{\partial s_y}, \frac{\partial V}{\partial s_z}\right) = \mathbf{0}. \tag{4}
$$

It is rational to expect a solution from the equilibrium equations. If that solution is unique the equilibrium placement is stable, whereas in the case the solution is not unique the placement is unstable. With non-unique solutions we mean multiple solutions (bifurcation case) or no solution at all (dynamic buckling).

3. Bifurcation analysis

Adopting *delay convention* for stability, Gilmore (1981, p. 143), the system remains in a stable or metastable equilibrium place until that state disappears. Delay convention is used for studying mainly structural systems (Thompson and Hunt, 1973; Troger and Steindl, 1991). Nevertheless, Maxwell's convention demands for the system to follow the places of the global minima of the total potential, Gilmore (1981, p.143). Maxwell's convention for stability is used for studying material instabilities such as coexistence of phases phenomena, twinning of crystals etc. (Ericksen, 1991; Pitteri and Zanzotto, 2003). The analysis that follows conforms with the delay convention for stability.

Let us consider the generalized vector,

$$\mathbf{v} = (s_x, s_y, s_z) \tag{5}$$

defining the placement of the system and the loading vector $\mathbf{\Lambda}$ of the loading parameters,

$$\mathbf{\Lambda} = (T_x, T_y, T_z), \tag{6}$$

then the total potential energy function V may be expressed by

$$V = V(\mathbf{v}, \mathbf{\Lambda}). \tag{7}$$

If a position $(\mathbf{v}^0, \mathbf{\Lambda}^0)$ is an equilibrium placement, the equilibrium path in the neighborhood,

$$\mathbf{v} = \mathbf{v}^0 + \mathbf{dv},\ \ \mathbf{\Lambda} = \mathbf{\Lambda}^0 + \mathbf{d\Lambda} \tag{8}$$

satisfies Taylor's expansion of the equilibrium equation,

$$\mathbf{V}_1(\mathbf{v}, \mathbf{\Lambda}) = \mathbf{V}_1^0 + \mathbf{V}_2^0\mathbf{dv} + \frac{\mathbf{V}_3^0}{2!}\mathbf{dv}^2 + \frac{\mathbf{V}_4^0}{3!}\mathbf{dv}^3 + \dot{\mathbf{V}}_1^0\mathbf{d\Lambda} + \mathbf{o}(|\mathbf{dv}|^3 + |\mathbf{d\Lambda}|) = \mathbf{0} \tag{9}$$

with $\mathbf{V}_1 = \nabla V$, $\mathbf{V}_2 = \nabla\nabla V$, $\mathbf{V}_3 = \nabla\nabla\nabla V$ and $\dot{\mathbf{V}} = \frac{\partial V}{\partial \mathbf{\Lambda}}$ and the upper-script indicating evaluation at the placement $(\mathbf{v}^0, \mathbf{\Lambda}^0)$. The symbol $\mathbf{o}(|\mathbf{dv}|^3 + |\mathbf{d\Lambda}|)$ means truncation of the higher order terms than the ones included in the parenthesis. Since the terms included in Eq. (9) should be of the same order of magnitude, the higher order terms are important in specific cases. Recalling that $(\mathbf{v}^0, \mathbf{\Lambda}^0)$ is an equilibrium placement, $\mathbf{V}_1^0 = 0$ and Eq. (9) yields,

$$\mathbf{V}_2^0\mathbf{dv} + \dot{\mathbf{V}}_1^0\mathbf{d\Lambda} = \mathbf{0}. \tag{10}$$

Hence, if

$$\mathbf{V}_2^0 = \begin{bmatrix} \dfrac{\partial^2 V^0}{\partial s_x^2} & \dfrac{\partial^2 V^0}{\partial s_x \partial s_y} & \dfrac{\partial^2 V^0}{\partial s_x \partial s_z} \\ \dfrac{\partial^2 V^0}{\partial s_y \partial s_x} & \dfrac{\partial^2 V^0}{\partial s_y^2} & \dfrac{\partial^2 V^0}{\partial s_y \partial s_z} \\ \dfrac{\partial^2 V^0}{\partial s_z \partial s_x} & \dfrac{\partial^2 V^0}{\partial s_z \partial s_y} & \dfrac{\partial^2 V^0}{\partial s_z^2} \end{bmatrix} \tag{11}$$

is not a singular matrix, the equilibrium Eq. (10) accepts a unique solution,

$$\mathbf{dv} = -\left(\mathbf{V}_2^0\right)^{-1}\dot{\mathbf{V}}_1\mathbf{d\Lambda}. \tag{12}$$

In the case $\mathbf{V}_2^0$ is a singular matrix yielding,

$$\det \mathbf{V}_2^0 = 0 \tag{13}$$

the critical conditions $\mathbf{\Lambda}^0 = (T_x^0, T_y^0, T_z^0)$ have been reached, corresponding to the critical placement. In that case a single vector direction $\mathbf{dt}$ for the simple bifurcation case, or two vector directions $\mathbf{dt}$ for the compound bifurcation case, satisfies the equation,

$$\mathbf{V}_2^0\mathbf{dt} = 0. \tag{14}$$

4. Simple bifurcation case

The incremental vector is defined, in this case, by

$$\mathbf{dv} = \zeta \mathbf{dt}, \tag{15}$$

where ζ is a parameter. Hence, the equilibrium Eq. (10) yields,

$$\zeta^2 = -\frac{2\dot{\mathbf{V}}_1 \mathbf{d\Lambda dt}}{\mathbf{V}_3(\mathbf{dt})^3} \tag{16}$$

if $\mathbf{V}_3(\mathbf{dt})^3 \neq \mathbf{0}$. This case corresponds to fold catastrophe. In case $\zeta^2 > 0$ two equilibrium placements are possible. In case $\zeta^2 < 0$ no equilibrium placement is possible and motion is expected. In the case $\mathbf{V}_3(\mathbf{dt})^3 = \mathbf{0}$ the equilibrium Eq. (12) yields,

$$\zeta^3 = -\frac{6\zeta \dot{\mathbf{V}}_2 \mathbf{dt}^2 \mathbf{d\Lambda}}{\mathbf{V}_4 \mathbf{dt}^4 + 3\mathbf{V}_3 \mathbf{dt}^2 \mathbf{b}} \tag{17}$$

corresponding to the cusp singularity with $\mathbf{b}$ a vector defined by the relation,

$$\mathbf{V}_3(\mathbf{dt})^2 + \mathbf{V}_2 \mathbf{b} = \mathbf{0},$$

see Lazopoulos and Markatis (1995).

5. Compound bifurcation of the six-strut tensegrity model

In the present case, the singular matrix $\mathbf{V}_2^0$ includes entries of the same value equal to ϑ. Therefore, the kernel $\mathbf{a}_2$ of the singular matrix $\mathbf{V}_2^0$ satisfying the equation,

$$\mathbf{V}_2^0 \mathbf{a}_2 = \mathbf{0} \tag{18}$$

with the kernel $\mathbf{a}_2$ may be described by

$$\mathbf{a}_2 = [\varphi_1 \quad \varphi_2] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}. \tag{19}$$

Standard bifurcation methods require elimination of passive coordinates and normalization of the total potential energy function. However, Lazopoulos (1994) and Lazopoulos and Markatis (1995) have presented a free coordinate branching approach with classification of the various singularities. Following that procedure, the tangent to the bifurcating equilibrium branch is described by a vector $\mathbf{a}_1$ in R^2 with $\mathbf{a}_1 = [\delta_1 \quad \delta_2]$, and

$$\begin{bmatrix} ds_x \\ ds_y \\ ds_z \end{bmatrix} = \zeta \mathbf{a}_2 \mathbf{a}_1 = \zeta[\delta_1 \varphi_1 \quad \delta_2 \varphi_2]. \tag{20}$$

Further, the vector $\mathbf{a}_1 \varepsilon R^2$ is defined by the existence of a vector $\boldsymbol{\gamma} = [\gamma_1 \quad \gamma_2]$ satisfying the equations,

$$\begin{aligned} &\mathbf{V}_3^0(\mathbf{a}_2\mathbf{a}_1)^2(\mathbf{a}_2\boldsymbol{\gamma}) = \mathbf{0}, \\ &\dot{\mathbf{V}}_2^0(\mathbf{a}_2\mathbf{a}_1)(\mathbf{a}_2\boldsymbol{\gamma})\mathbf{d\Lambda} = \mathbf{0}. \end{aligned} \tag{21}$$

Eliminating γbetween the Eq. (21) a vector $\mathbf{a}_1$ is defined, describing the tangent vectors to the equilibrium paths. Three different solutions (vectors $\mathbf{a}_1$) of the system (21) classify the singularity as elliptic umbilic, whereas the unique solution classifies the singularity as hyperbolic and the two different vectors $\mathbf{a}_1$ classify the singularity as parabolic umbilic. Yet the parameter ζ may be found by the equilibrium Eq. (16), see Lazopoulos and Markatis (1995).

6. Applications

Two applications will be studied in the present section. The first application deals with the simple branching of the equilibrium of the six-strut tensegrity model with forces acting along one direction. The second application deals with the compound branching.

6.1. Simple bifurcation application

Let us consider the cytoskeletal tensegrity model with the strut length $L_0 = 4.97$. The applied force is directed only along the x-axis. Furthermore, the strain energy function is defined by,

$$W_i = \frac{1}{2}\left(\frac{l_i}{l_n} - 1\right)^2 - \frac{5.56}{6}\left(\frac{l_i}{l_n} - 1\right)^3 + \frac{4.12}{24}\left(\frac{l_i}{l_n} - 1\right)^4. \tag{22}$$

Therefore, equilibrium Eq. (4) yields,

$$\begin{aligned} &\frac{\partial W_1}{\partial s_x} + \frac{\partial W_2}{\partial s_x} + \frac{\partial W_3}{\partial s_x} - \frac{T_x}{8} = 0, \\ &\frac{\partial W_1}{\partial s_y} + \frac{\partial W_2}{\partial s_y} + \frac{\partial W_3}{\partial s_y} = 0, \\ &\frac{\partial W_1}{\partial s_z} + \frac{\partial W_2}{\partial s_z} + \frac{\partial W_3}{\partial s_z} = 0. \end{aligned} \tag{23}$$

The system of equilibrium equations consists of two equations due to the symmetry in the y and z directions. Likewise, the critical condition is defined by,

$$\det \mathbf{V}_2^0 = 0. \tag{24}$$

A critical placement (s_x, s_y, s_z) satisfying both Eqs. (23) and (24) may be found using the Mathematica computerized algebra pack. Indeed, if

$$H = \left(\frac{\partial W_1}{\partial s_y} + \frac{\partial W_2}{\partial s_y} + \frac{\partial W_3}{\partial s_y}\right)^2 + \left(\frac{\partial W_1}{\partial s_z} + \frac{\partial W_2}{\partial s_z} + \frac{\partial W_3}{\partial s_z}\right)^2 + (\det \mathbf{V}_0^2)^2 \tag{25}$$

a solution to the system (23) and (24) may be located looking for the minimum of the function H to be equal to zero. Indeed, with the natural (initial) length $l_n = 2.72$ a critical state may be located with $s_x = 5.0$, $s_y = 5.76$, $s_z = 5.79$. That critical placement is reached with $T_x = 0.0687$. The Hessian matrix at the critical placement, see Eq. (11), has been computed and found equal to,

$$\mathbf{V}_2^0 = \begin{bmatrix} -0.060 & -0.0006 & -0.012 \\ -0.0006 & -0.141 & -0.012 \\ -0.012 & -0.012 & -0.133 \end{bmatrix}. \tag{26}$$

The vector $\mathbf{dt} = [\, dt_1 \quad dt_2 \quad dt_3 \,]^{\mathrm{T}}$ satisfying Eq. (14), i.e. $\mathbf{V}_2^0\mathbf{dt} = \mathbf{0}$, may be given by,

$$\mathbf{dt} = [\, 1 \quad 0.42 \quad -4.93 \,]^{\mathrm{T}}. \tag{27}$$

Proceeding to the prescription of the post-critical equilibrium path, Eq. (16) defines the parameter ζ. Indeed,

$$\begin{aligned}\mathbf{V}_3^0(\mathbf{dt})^3 &= \frac{\partial^3 V}{\partial s_x^3}dt_1^3 + 3\frac{\partial^3 V}{\partial s_x \partial s_y^2}dt_1 dt_2^2 + 3\frac{\partial^3 V}{\partial s_x \partial s_z^2}dt_1 dt_3^2 + \frac{\partial^3 V}{\partial s_y^3}dt_2^3 + 3\frac{\partial^3 V}{\partial s_y \partial s_x^2}dt_2 dt_3^2 + 3\frac{\partial^3 V}{\partial s_y \partial s_z^2}dt_2 dt_3^2 + \frac{\partial^3 V}{\partial s_z^3}dt_3^3 \\ &\quad + \frac{\partial^3 V}{\partial s_z^3}dt_3^3 + 3\frac{\partial^3 V}{\partial s_z \partial s_x^2}dt_3 dt_1^2 + 3\frac{\partial^3 V}{\partial s_z \partial s_y^2}dt_3 dt_2^2 + 3\frac{\partial^3 V}{\partial s_z \partial s_y \partial s_z}dt_1 dt_2 dt_3 \\ &= 13.23\end{aligned}$$

$$\dot{\mathbf{V}}_1\mathbf{dt}\mathbf{d}\boldsymbol{\Lambda} = -dT_x$$

Hence, according to Eq. (16),

$$\zeta^2 = -\frac{2\dot{\mathbf{V}}_1\mathbf{d}\boldsymbol{\Lambda}\mathbf{dt}}{\mathbf{V}_3(\mathbf{dt})^3} = 0.151 dT_x. \tag{28}$$

Consequently, for $dT_x > 0$ the equilibrium placements are,

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} 5.0 + \zeta \\ 5.76 + \zeta 0.42 \\ 5.79 - \zeta 4.93 \end{bmatrix}.$$

6.2. Compound branching application

Let us consider, in the present case, the six-strut tensegrity model with the strut length of the perfect system $L_0 = 5$. However initial imperfections may be present at the lengths of the struts and $L_{\mathrm{I}} = L_0 + e$, $L_{\mathrm{II}} = L_0 + 2e$, $L_{\mathrm{III}} = L_0 + 3e$. Let us point out that the solution to the problem requires existence of some imperfections. Those imperfections may also be introduced by the elastic coefficients of the strain energy function. The applied force system is directed only along the three axes with equal magnitude. In this case the $s_x = s_y = s_z$. Furthermore, the strain energy function is defined by,

$$W_i = \frac{1}{2}\left(\frac{l_i}{l_{\mathrm{n}}} - 1\right)^2 - \frac{2}{6}\left(\frac{l_i}{l_{\mathrm{n}}} - 1\right)^3. \tag{29}$$

Therefore, equilibrium Eq. (4) yields,

$$\frac{\partial W_1}{\partial s_i} + \frac{\partial W_2}{\partial s_i} + \frac{\partial W_3}{\partial s_i} - \frac{T_i}{8} = 0, \quad i = x, y, z. \tag{30}$$

Further, the Hessian $\mathbf{V}_2^0$ is singular with equal entries iff,

$$\frac{\partial^2 W_1}{\partial s_x^2} + \frac{\partial^2 W_2}{\partial s_x^2} + \frac{\partial^2 W_3}{\partial s_x^2} = \frac{\partial^2 W_1}{\partial s_x \partial s_y} + \frac{\partial^2 W_2}{\partial s_x \partial s_y} + \frac{\partial^2 W_3}{\partial s_x \partial s_y}. \tag{31}$$

A critical placement of compound bifurcation is defined by a solution of the system (30) and (31) if the natural (initial) length of the cables is $l_{\mathrm{n}} = 2.5$. Indeed, the system (30) and (31) yields,

$$s_x = s_y = s_z = 6.616. \tag{32}$$

For a tensegrity system with the parameters,

$$\alpha = 1, \quad \beta = -2, \quad c = 0, \quad L_0 = 5, \quad l_n = 2 \tag{33}$$

the perfect system ($e = 0$) yields

$$x_0 = y_0 = z_0 = 6.616, \quad T_x^0 = T_y^0 = T_z^0 = 0.334. \tag{34}$$

Eq. (21a) yields for $\boldsymbol{\gamma} = (\gamma_1 \quad \gamma_2)$, $\gamma = \gamma_1/\gamma_2$, $\delta = \delta_1/\delta_2$

$$-0.075\delta^2 + 0.0716\delta\gamma - 0.051\gamma^2 = 0, \tag{35}$$

whereas, Eq. (21b) yields, assuming $\boldsymbol{\Lambda} = (T_x, T_y, T_z, e)$

$$(-0.7334 + 0.362\delta + 0.36\gamma - 0.74\gamma\delta)e = 0. \tag{36}$$

The common real solution of the system (35), (36) is,

$$\delta = 1.02, \quad \gamma = -0.92 \tag{37}$$

and the singularity is hyperbolic umbilic.

Therefore, the post-critical path is described by the field,

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} 6.616 + \xi \\ 6.616 \\ 6.616 - \xi \end{bmatrix}. \tag{38}$$

The post-critical, loading in addition, may be given by,

$$T_x = 0.334 + \tau_1, \quad T_y = 0.334 + \tau_2, \quad T_z = 0.334 + \tau_3. \tag{39}$$

The total potential $V = V(\xi, e, \tau_1, \tau_2, \tau_3)$ may be computed. Then the equilibrium equation is defined by

$$\frac{\partial V}{\partial \xi} = 0 \tag{40}$$

yielding,

$$\xi = \frac{\tau_3 - \tau_1}{0.15} e^{-1}. \tag{41}$$

Therefore, Eqs. (38) and (41) yield the post-critical equilibrium path.

7. The strut buckling

In this case, the axial compressive load on the struts AA or $A'A'$ has reached the Euler critical value,

$$P_0 = \left(\frac{\pi}{L}\right)^2 EI, \tag{42}$$

with EI the bending stiffness of the struts AA or $A'A'$ and L_0 the initial length of the struts. It is also assumed that the compressive loads P_B, P_C of the struts BB or $B'B'$ and CC or $C'C'$ respectively are less than the corresponding Euler critical loads. This is not a restriction, since the compressive loads are different. The compressive loads in Coughlin and Stamenovic (1997) are equal because no external loading is applied and the initial equilibrium placement is critical. Here the critical placement shows up after the application of the load T_x. Due to the buckling of the AA or $A'A'$ struts, the deflection of the strut is defined by,

$$w(s) = \zeta \sin \frac{\pi s}{L}, \tag{43}$$

with s the arc-length of the inextensible strut and the chord-lengths of the struts are equal to,

$$\begin{aligned} L_{\mathrm{I}} &= L - \frac{1}{2}\int_0^L \left((w'(s))^2 - \frac{(w'(s))^4}{12} \right) ds = L + \frac{-16L^2\zeta^2\pi^2 + 16\zeta^4\pi^4}{64L^3}, \\ L_{\mathrm{II}} &= L_{\mathrm{III}} = L. \end{aligned} \tag{44}$$

With the application of the load T_x only, see Fig. 1, the total potential energy function is expressed by,

$$V = 8(W_1 + W_2 + W_3) + 2\int_0^L \left(\frac{EI}{2}\frac{1}{\rho^2} \right) ds - T_x s_x, \tag{45}$$

where $\frac{1}{\rho}$ is the curvature of the elastic curve of the inextensible struts AA and $A'A'$ with $|w| \ll 1$ and $w' = \frac{dw}{ds}$ is the angle between the elastic curve and the axis of these struts.

Likewise, the non-linear curvature is approximated by,

$$\frac{1}{\rho} \approx w''(s)\left(1 + \frac{1}{2} w'(s)^2 \right). \tag{46}$$

Consequently the potential energy function V depends on the variables s_x, s_y, s_z, ζ.

Hence, the buckling of the struts AA and $A'A'$ problem is formulated by the equilibrium equations,

$$\frac{\partial V}{\partial s_x} = \frac{\partial V}{\partial s_y} = \frac{\partial V}{\partial s_z} = 0 \quad \text{at } \zeta = 0. \tag{47}$$

Eq. (47) are equivalent to the static equations of Coughlin and Stamenovic (1997),

$$T_x = 2F_1 \frac{s_x - L}{l_1} + 2F_2 \frac{s_x}{l_2}, \tag{48}$$

$$F_1 \frac{s_y}{l_1} = F_3 \frac{L - s_y}{l_3}, \tag{49}$$

$$F_2 \frac{L_{\mathrm{I}} - s_z}{l_2} = F_3 \frac{s_z}{l_3}. \tag{50}$$

The critical condition for the buckling of the struts AA' has been reached when

$$\frac{\partial^2 V}{\partial \zeta^2} = 0. \tag{51}$$

Simple algebra reveals that the critical condition (51) is equivalent to that the critical condition has been reached when the compressive load P_A equals to the Euler buckling load for a simply supported beam. Thus Eq. (51) yields the Euler critical P_{Ac},

$$P_{Ac} = \left(\frac{\pi}{L} \right)^2 EI, \tag{52}$$

where P_A is the compressive load on the strut AA' given by

$$P_A = F_1 \frac{L_{\mathrm{I}}}{l_1} + F_2 \frac{L_{\mathrm{I}} - s_z}{l_2} \tag{53}$$

see Coughlin and Stamenovic (1997). In addition the compressive loads on the struts BB' and CC' are given by,

$$P_B = F_3 \frac{L_{\text{II}}}{l_3} + F_1 \frac{L_{\text{II}} - s_x}{l_1}, \tag{54}$$

$$P_C = F_2 \frac{L_{\text{III}}}{l_2} + F_3 \frac{L_{\text{III}} - s_y}{l_3}. \tag{55}$$

It is assumed, evidently, that the compressive load P_A is higher than the other compressive loads P_B and P_C. This assumption needs further investigation.

Implementing the theory, we consider a six-strut tensegrity model with the parameters,

$$\alpha = 7, \quad \beta = -5.59, \quad c = 0, \quad L = 5.5, \quad l_0 = 1, \quad EI = 1.085. \tag{56}$$

A critical placement with buckling of the struts may be located with the help of the Mathematica computerized pack. Indeed, forming a function

$$H = \left(\frac{\partial V}{\partial s_x}\right)^2 + \left(\frac{\partial V}{\partial s_y}\right)^2 + \left(\frac{\partial V}{\partial s_z}\right)^2 + \left(\frac{\partial^2 V}{\partial \zeta^2}\right)^2 \tag{57}$$

we are looking for a placement with minimum of the function $H = 0$. With the FindMinimum program of the Mathematica, a critical equilibrium placement $(s_x^0,\ s_y^0,\ s_z^0,\ \zeta = 0)$ may be located with min $H = 0$. Indeed, a critical placement may be located with,

$$s_x^0 = 3.78, \quad s_y^0 = 3.94, \quad s_z^0 = 3.94, \quad T_x^0 = 1. \tag{58}$$

Increasing the load so that $T_x = T_x^0(1+\lambda)$ and $0 < \lambda \ll 1$, the equilibrium placement is defined by the variables

$$s_x = s_x^0 + s_x^1, \quad s_y = s_y^0 + s_y^1, \quad s_z = s_z^0 + s_z^1. \tag{59}$$

The incremental variables s_x^1, s_y^1, s_z^1 are defined by the equations,

$$\begin{aligned}
&\frac{\partial^2 W}{\partial s_x^{02}} s_x^1 + \frac{\partial^2 W}{\partial s_x^0 \partial s_y^0} s_y^1 + \frac{\partial^2 W}{\partial s_x^0 \partial s_z^0} s_z^1 - 2T_x^0 \lambda = 0, \\
&\frac{\partial^2 W}{\partial s_y^0 \partial s_x^0} s_x^1 + \frac{\partial^2 W}{\partial s_y^{02}} s_y^1 + \frac{\partial^2 W}{\partial s_y^0 \partial s_z^0} s_z^1 = 0, \\
&\frac{\partial^2 W}{\partial s_z^0 \partial s_x^0} s_x^1 + \frac{\partial^2 W}{\partial s_z^0 \partial s_y^0} s_y^1 + \frac{\partial^2 W}{\partial s_z^{02}} s_z^1 = 0,
\end{aligned} \tag{60}$$

where all the quantities have been computed at the critical placement. Solving the system with the help of Mathematica we get,

$$s_x^1 = -0.3946\lambda, \quad s_y^1 = -0.229\lambda, \quad s_z^1 = -0.223\lambda. \tag{61}$$

Substituting s_x, s_y, s_z from Eq. (59) into the potential function V, Eq. (45) the equilibrium equation

$$\frac{\partial V}{\partial \zeta} = 0 \tag{62}$$

yields,

$$3.49\zeta^3 + 0.8360\zeta\lambda = 0. \tag{63}$$

Eq. (63) yields for $\lambda \geqslant 0$ no non-zero real solutions. Therefore, motion is expected, since equilibrium ceases to exist. Therefore, the critical Euler buckling load does not assure bifurcation, since dynamic response may arise.

8. Conclusion-further studies

Stability studies for a six-strut tensegrity model have been performed. Simple and compound bifurcation states have been studied. The stability of the six-strut model with buckling of the one pair of the struts has also been worked out. Critical conditions have been located and post-critical states have been described. It is pointed out that stable bifurcation does not always exists when the struts buckle. Compound buckling has also been discussed. The theory was implemented to various applications.

As it has been pointed out, the present study deals with stability studies adopting the (local) delay convention for stability. Stability studies adopting Maxwell's convention for stability will be also quite interesting. Adopting the global (Maxwell's) convention for stability, coexistence of phases phenomena may show up along the cables. Cables with smart materials may exhibit that behaviour. Furthermore, experimental studies may reveal the proper stability convention for tensegrity structures.

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